Probability I

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What is Probability Theory?

Probability theory is the branch of mathematics concerning numerical descriptions of how likely an event is to occur or how likely it is for a proposition to be true.

Some examples of questions answered using probability theory are:

- What is the probability that if I roll two dice, the sum of the results will add up to 8?
- If I flip a coin 100 times, what is the probability that I will get 6 tails in a row sometime in the sequence?
- What is the expected 1-Yr return of my portfolio?
- What is the probability that my $TSLA call option will expire in the money?

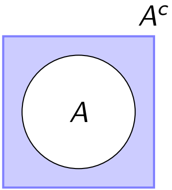
A Brief Primer on Set Theory

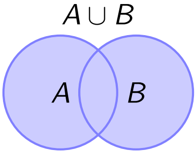
A **set** is a collection of objects. If we want to say that an object a belongs to the set S, we write $a \in S$.

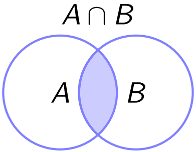
A set R is called a **subset** of another set S if every element of R also belongs to S. This is written as $R \subset S$.

The **complement** of a set S, denoted S^c, is the set of all things that are not in S.

The **union** of two sets S and R, denoted $S \cup R$, is the set of all things that are in S *or* R. The **intersection** of two sets S and R, denoted $S \cap R$, is the set of all things that are in S *and* R.







The Axioms of Probability

The set of all possible outcomes of an experiment is called the **sample space**, denoted S. Any subset $E \subset S$ is called an **event**. The **probability** of an event E is written as $\mathbb{P}(E)$.

Two events E and F are said to be **mutually exclusive** if their intersection $E \cap F$ is empty. An example of this would be if E was a coin landing on heads and F was a coin landing on tails. Here, $E \cap F = \emptyset$ (empty set).

Axioms

(a) For any event $E \subset S$,

$$0 \leq \mathbb{P}(E) \leq 1$$

(b) $\mathbb{P}(S) = 1$

(c) For a collection of mutually exclusive events $E_1, E_2, E_3, \ldots,$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Permutations

If we are given a list of n objects, we define a **permutation** of this list to be an arragnement of the list.

The number of permuation of n objects is:

$$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n!$$

Example

Say we are given $\{a, b, c\}$. The list of permutations is:

abc	bac	cab
acb	bca	cba

There are $3! = 6$ permutations of 3 elements.

Combinations

If we are given a list of n objects, we define a **combination** to be a selection of items from the list such that the order does not matter (as opposed to permuations, where order *does* matter). The number of combinations of k-combinations of n objects is:

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

Example

Say we are given $\{a, b, c, d\}$. The list of 3-combinations is:

$abc \qquad abd \qquad acd \qquad bcd$

There are $\binom{4}{3} = 4$ 3-combinations of 4 elements.

Equally Likely Outcomes

In many experiments, we can assume that all outcomes in the sample space are equally likely to occur. In this case:

$$\mathbb{P}(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$

Example

Suppose I have 5 white balls and 6 black balls in a bag. What is the probability that if I draw three balls from the bag, one will be white and two will be black?

$$\mathbb{P}\left(1 \text{ white and } 1 \text{ black}\right) = \frac{\binom{5}{1}\binom{6}{2}}{\binom{11}{3}} = \frac{5}{11}$$

So, the probability that you draw one white and three black balls is $\frac{5}{11}$.
Note: This doesn't say anything about the order that they will be drawn.

Probability Laws

Here are some useful identities to know when working with probabilities:

Probability of Unions

$$\mathbb{P}(X \cup Y) = \mathbb{P}(X) + \mathbb{P}(Y) - \mathbb{P}(X \cap Y)$$

The last term, $\mathbb{P}(X \cap Y)$, is there to ensure that we aren't double-counting the overlap (this would be like the middle of a Venn Diagram being counted twice).

Probability of Intersections of Independent Events

$$\mathbb{P}(X \cap Y) = \mathbb{P}(X) \cdot \mathbb{P}(Y)$$

Conditional Probability

The **conditional probability** that an event E occurs, given that an event F has occured, is written as: $\mathbb{P}(E \mid F)$. A useful formula for evaluating this, given that $\mathbb{P}(F) > 0$, is:

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$$

It can be useful to rearrange this equation into:

$$\mathbb{P}(E \cap F) = \mathbb{P}(E \mid F)\mathbb{P}(F)$$

We can generalize the above into what is called the **multiplication rule**:

$$\mathbb{P}(E_1 \cap E_2 \cap \cdots \cap E_n) = \mathbb{P}(E_1)\mathbb{P}(E_2 \mid E_1)\mathbb{P}(E_3 \mid E_1 \cap E_2) \cdots \mathbb{P}(E_n \mid E_1 \cap \cdots \cap E_{n-1})$$

Bayes' Theorem

Bayes' theorem is a very important result in probability theory that allows us to more easily work with conditional probabilities.

Bayes' Theorem

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

This theorem is very useful when we are trying to rewrite an unknown probability in terms of probabilities that we know.

Recap

In this presentation, we have laid out the fundamentals of probability theory.

We spoke about the fundamentals of counting (called *combinatorial analysis*). We also constructed the axioms and basic laws of probability.

Next time, using this foundation, we will discuss *random variables*.